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Identification of nonlinear parameters for reduced order models

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Abstract

Constructing nonlinear structural dynamic models is a goal for numerous research and development organizations. Such a predictive capability is required in the development of advanced, high-performance aircraft structures. Specifically, the ability to predict the response of complex structures to engine induced and aero acoustic loading has long been a United States Air Force (USAF) goal. Sonic fatigue has plagued the USAF since the advent and adoption of the turbine engine. While the problem has historically been a maintenance one, predicting the dynamic response is crucial for future aerospace vehicles. Decades have been spent investigating the dynamic response and untimely failure of aircraft structures, yet little work has been accomplished towards developing practical nonlinear prediction methods. Further, the last decade has witnessed an appreciable amount of work in the area of nonlinear parameter identification. This paper outlines a unique and important extension of a recently introduced nonlinear identification method: Nonlinear Identification through Feedback of the Outputs (NIFO). The novel extension allows for a ready means of identifying nonlinear parameters in reduced order space using experimental data. The nonlinear parameters are then used in the assembly of reduced order models, thus providing researchers with a means of conducting predictive studies prior to expensive and questionable experimental efforts. This paper details both an analytical and experimental study conducted on a well-characterized clamped–clamped beam subjected to broadband random loading. Amplitude dependent, constant stiffness parameters were successfully identified for a multiple-degree-of-freedom (MDOF) nonlinear reduced order model. The nonlinear coefficients identified from the analytical scenario compare well with previously published studies of the beam. Nonlinear parameters were also successfully identified from the raw experimental data. Finally, a MDOF nonlinear reduced order model, constructed from experimental data, was used to predict the experimental response of the beam to other loading conditions. Beam response spectra and average displacement values from the prediction model also compare well with the experimental results.

1. Introduction

Sonic fatigue has long been a maintenance issue for the United States Air Force (USAF). With the advent of more powerful computers and more efficient computing methods, there have been numerous attempts to provide research tools capable of predicting the response of aircraft structures to severe acoustic loading. One

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of the key assumptions of any analytical based structural prediction method is the ability to successfully model the structure. If the structure is well characterized or idealized, then prediction *can* be a relatively simple task. Given the complexity of future aerospace structures, confidence solely in modeling is suspect at best. This confidence in analytical or finite element (FE) modeling is further reduced given unknown boundary conditions and suspect material properties. What is required is a robust methodology for the identification of reduced order models. This is important for several reasons; first for obvious prediction purposes and second for the purposes of FE model updating. The ability to provide useful, experimentally derived parameters as input to FE methods is quite powerful. The method presented in this study also allows for rapid modification of the assumed nonlinear form, thus providing researches with a means of identifying a full spectrum of reduced order nonlinear models—parametric or not. This is a significant difference from many of the methods in the literature, where the identification of simultaneous nonlinearities or the quick adoption of various forms is prohibitive. Another key feature of the modified Nonlinear Identification through Feedback of the Outputs (NIFO) method presented here is the ability to identify the nonlinear parameters directly with no iteration or presumptions as to appropriate initial values. Also, the method is free from restrictive loading conditions, i.e., harmonic balance or the method of multiple scales. In fact, the results presented here are based upon broadband random inertial loading. Two of the underlying assumptions in the method are first, the nonlinearities are indeed exercised and second, an underlying or nominal linear system exists. Both of these assumptions are indeed satisfied for the typical aerospace structures experiencing acoustic fatigue, i.e., stiffened thin skin aircraft structures.

The research and development of nonlinear parameter identification methods is a rapidly growing area. This recent growth is due to a need for greater understanding of the nonlinear response of systems as previously described. Some of the more recent developments follow, particularly those concerned with reduced order modeling. A survey of developing methods yields various approaches to address the problem; purely analytical methods, hybrid approaches combining reduced order and physical space models or hybrid analytical and experimental results, and finally, purely experimental based methods. This review will start with a discussion of the purely analytical approaches. The idea of generating reduced order models based solely on FE results has been studied extensively by Nash [1], Shi and Mei [2], McEwan et al. [3–5], Hollkamp et al. [6,7], Mignolet et al. [8], Przekop and Rizzi [9], and Muravyov and Rizzi [10]. In all of these cases, nonlinear reduced order models are assembled using analytically identified or derived nonlinear coefficients, thus allowing for computationally tractable prediction models. This idea of reduced order models has seen recent interest in order to address a problem common to aircraft designers faced with nonlinear dynamical issues. To model the response of a realistic aircraft structure to realistic loading requires thousands if not millions of FE degrees of freedom (DOF). The successful nonlinear dynamic analysis of such a model is questionable, even with high-performance computers. What is required is a computationally efficient means of analyzing these structures. The intent of Nash [1] and Shi and Mei [2] was to arrive at nonlinear reduced order models directly through the manipulation of the FE stiffness matrices. In particular, Shi and Mei [2] use only bending modes in their basis set to describe the response of beam and plate structures. Their in-plane DOFs are described in terms of the transverse ones and thus are condensed. The resulting geometric nonlinear coefficients are therefore directly evaluated. Two issues preclude widespread use of this approach; first, the condensation approach is applicable only to structures whereby in-plane and transverse DOFs are separable, and second, it is doubtful that commercial FE algorithms could be utilized, at least easily, as the method requires direct modification and manipulation of the stiffness matrices. The basic idea common to the remaining methods has focused on using generic FE algorithms to identify constant, nonlinear stiffness coefficients indirectly. In order to capitalize on the convenience and flexibility of commercial FE algorithms, McEwan [3–5] devised a means of identifying geometric nonlinear coefficients from a series of nonlinear FE static analyses. Using those identified coefficients and the results of FE eigensolutions, a reduced order model can be assembled, and thus providing engineers with a viable tool, i.e., the computational expense of such a reduced order model is negligible when compared with high-fidelity codes.

Hollkamp et al. [6] compared the results of a well-characterized clamped–clamped beam experiment to the results of reduced order models generated using various FE based methods. Hollkamp [7] also presented an in-depth discussion of FE derived nonlinear reduced order modeling methods. In both references, locally developed algorithms were compared to those developed by McEwan [3–5], Mignolet et al. [8], Przekop and

Rizzi [9] and Muravyov and Rizzi [10]. Comparisons were made with the dynamic response generated using an FE model of the same clamped beam, albeit with idealized boundary conditions. The nonlinear coefficients of the various methods, for both single and two DOF models were presented. Some of the principle issues uncovered in the discussion include the appropriate selection or more accurately, the appropriate assumptions when assembling a basis set for this type of problem. All of the previously mentioned methods approach the problem from a slightly different perspective, but all attempt to capture the sonic fatigue type response of thin skinned aircraft structures, or the modeling of the in-plane motion through an appropriate basis. It was determined that the method of McEwan [3–5] is the most appropriate for this class of problems, as there is no distinction between the in-plane and transverse basis vectors. The in-plane motion of beams or panels is captured through an implicit condensation similar to the method previously discussed. This is an important point, as a similar approach is presented here, i.e., implicit condensation of the in-plane or membrane basis vectors, is conducted via experimental identification. A limitation with these reduced order analytical methods is in the tractable form of the assumed nonlinearities. In other words, they are necessarily limited to a stiffness dependent or static form in lieu of computationally expensive nonlinear full-model dynamic analyses; the very analysis the researchers are attempting to avoid. For example, in the work of Hollkamp [6,7], a full suite of static nonlinear FE analyses is conducted to provide the data necessary for the respective identification methodology. The intent of the analytical based methods is to assemble nonlinear reduced order models without having to rely on computationally expensive and oftentimes restrictive dynamic analyses. While the benefits of utilizing commercial FE algorithms to identify nonlinear parameters are obvious, the method presented in this study is not limited solely too static nonlinearities, but again, is adaptable to a variety of nonlinear forms. Thus, the method presented in this study is assumed to be complimentary to the analytical methods described herein.

A survey of the purely experimental based methods will now be presented. An excellent approach to nonlinear parameter identification for reduced order models is the one taken by Yasuda and Kamiya [11]. Yasuda begins with an elasticity approach and arrives at the governing equations in modal space and nonlinear in stiffness via the Galerkin Method. This proposed time-domain approach is divided into two complimentary methods. Their first approach is to take measured response data and solve implicitly for the basis set via a weighted eigenvalue problem, similar to a singular value decomposition approach. Then, linear and nonlinear modal parameters are identified using a linear least-squares approach. Their second approach is to identify all of the parameters including the basis set, in a single step. This is accomplished by the minimization of a form of Lagrange's energy equation. This approach requires an iterative scheme, as the coupled equations resulting from the minimization step are nonlinear. Initial conditions for the iteration are obtained from the results of the first method. Both analytical and experimental results were presented. Like the analytical work of the previous discussion, the experiment consisted of a beam, albeit in a pinned–pinned configuration. Measurements were recorded at four locations along the beam. Input for the experimental method was a fast sine-sweep, accomplished via a non-contacting magnetic excitation source. Excellent agreement was obtained between analysis and experiment using the second method detailed in the reference.

Richards and Singh [12,13] introduced the Conditioned Reverse Path (CRP) method, itself an extension of the Reverse Path Method of Bendat [14]. Although the method is not a reduced order one, it is a popular choice for nonlinear parameter identification as evidenced by its rather widespread use [15–18]. The CRP method is generalized to account for multiple-input, multiple-output (MIMO) scenarios, diagnosing nonlinearities away from the location of the applied force, as well as for the typical testing configuration of fewer inputs than measured response points. The CRP method is a spectral one, where the inputs and outputs of the measured system are transposed, thus considering the forces as outputs and the measured responses as combinations of correlated and uncorrelated input components. The method is also a multi-step process, where first the underlying linear spectra are obtained via a process of separating the measured responses and forces into correlated and uncorrelated components. Estimates of the nonlinear coefficients are obtained in a second step. Successful comparisons were made between experimental low-level FRFs and the underlying linear estimates using 'conditioned' H_1 and H_2 FRF estimates [12]. Further, Richards and Singh [13] use the CRP method to estimate nonlinear coefficients trying various polynomial trial functions to account for the nonlinear behavior. This is an important area of research, as the form of the nonlinearity is not always known a priori. It was noted that a unique description of the nonlinear form was not expected. This idea of a general nonlinear trial function was investigated later by Adams and Allemang [19] and Martell et al. [20], with the former study utilizing general orthogonal polynomial trial functions in the presence of unknown nonlinearities. Kerschen et al. [16] presents a study of the CRP method using experimental data for a well-characterized beam setup. Further, Kerschen and Golinval [18] use the CRP method for FE model updating purposes. It was noted that the CRP methodology requires additional development, particularly in the areas of identifying multiple nonlinearities and with the computationally intensive nature of the method [17].

Kerschen et al. [21,22] and Lenaerts et al. [23,24] present two nonlinear identification methods, the restoring force method and a formulation based upon proper orthogonal decomposition (POD), analogous to reduced order modeling. The restoring force surface method is essentially a non-parametric, single-degree-of-freedom (SDOF) method. The authors note that the method can be extended to a MDOF scenario, although some of the attributes of the method are lost in the extension. The authors present the results of the method applied to various analytic and experimental beam configurations, as well as a European benchmark experiment known as the 'VTT benchmark'. One of the interesting notes made in the paper, is the use of a significance factor [25]. This factor attempts to measure the significance of the retained nonlinear coefficients. Recall, the restoring force method is a non-parametric one and without a priori knowledge of the nonlinear form, some means is required to measure the effectiveness of the retained nonlinear terms. Recall also the work of Richards and Singh, where a general polynomial trial function was used to capture the effect of a nonlinear term in an analytical model. It is again important to point out that a unique solution was not possible, nor expected. Good SDOF comparisons are made with the various analytical and experimental cases. Interestingly enough, because the restoring force method is a SDOF one, the experimental beam data was filtered in order to retain only the response of the first mode. Further, the restoring force method requires information regarding the acceleration, velocity and displacement of the structure in question. Acceleration measurements were obtained as is typical in experimental dynamic analysis, followed by numerical integration using a trapezium rule to obtain the velocity and displacement time histories. These issues will be discussed subsequently in further detail. The POD methodology, as utilized by the authors, is an interesting one for several reasons. It is essentially an updating procedure, and therefore can be thought of as a hybrid analytical/experimental method. The method seeks to minimize the differences between measured and simulated data thus resulting in a mathematical representation of the structure. When comparing the CRP to the POD based method, the authors note the CRP method solely identifies nonlinearities and the underlying modal properties, while the POD based approach identifies the nonlinear terms and also results in a mathematical model useful for prediction. This issue and its relation to the present work will also be discussed subsequently.

Recently, Platten et al. [26,27] and Naylor et al. [28] applied their Nonlinear Resonant Decay Method (NL-RDM) to identify modal models for both analytical and experimental systems. The NL-RDM method is also a multi-step process, accomplished in the time domain. First, a general sense of the linear and nonlinear characteristics of the system in question is obtained. This is accomplished through a series of MIMO tests conducted at various input levels. Second, the modal properties of the nominal linear system are obtained via traditional modal testing methods. Of prime importance in this step is the estimation of the normal mode shapes or the basis intended for the transformation between physical and modal space. Next, the system is excited via a burst sine excitation on a mode-by-mode basis. The excitation is applied at a level appropriate enough to exercise the respective nonlinearities. Measured accelerations are integrated in the frequency domain to obtain velocity and displacement time histories. The nonlinear parameters are then identified on a mode-by-mode basis. Finally, a full nonlinear modal model is assembled. One of the assumptions in the methodology is the subjective truncation of any mode that does not *appear* to interact or have influence over other modes. While this assumption is in keeping with the idea of linear modal truncation, it should be noted that the response of a system is a function but not necessarily a linear superposition of all of the structural modes. This is particularly important for nonlinear modal analysis, for while a mode may not be perceived through visual analysis to contribute to the nonlinear aspects of the response at particular loading inputs, the issue of modal saturation can play a significant role in nonlinear systems [29].

In an extensive dissertation, Siller [30] recently developed among other things, a 'hybrid modal technique' (HMT). The method seeks to identify nonlinear FRFs via expressions combining linear modal components with first-order approximations of nonlinear parameters in the physical domain. Immediately obvious is the neglect of the sub/super-harmonics of the problem resulting from the first-order approximation. Siller begins the HMT formulation with a proportionally damped, nonlinear ordinary differential equation. Assuming a harmonic input and response, the system is transformed into the frequency domain. By pre-multiplying the system of equations by the system eigenvectors, the system is further transformed into modal space. The modal transformation is implicitly accomplished using nonlinear normal modes (NNM), a basis set typically considered independent except at the respective resonances. Now the nonlinearities of the system are implicitly described through the NNMs and the nonlinear terms originally considered in the nonlinear ordinary differential equation. The end result of the method is the calculation of the system nonlinear FRFs accomplished through a modified Newton–Raphson iteration approach, where first a trial function for the response is assumed, along with the assumed nonlinear form in physical space and then the nonlinear equations are iterated upon until a predetermined convergence criterion is satisfied. It appears that the method is best suited for a model-updating type of approach. It should be noted that the results of the method were compared with the results of a harmonic balance solution at or near resonance only. Solutions were obtained solely in the vicinity of resonances due to the computational burden of the proposed method. The impetus for this hybrid approach is twofold; first, because the solution of the proposed method in modal space involves coupled nonlinear and therefore computationally prohibitive equations. It will be demonstrated that this is not the case for the identification of nonlinear coefficients given the structure considered in this work. The second reason mentioned for the pursuit of this hybrid approach, is that the condensation of the nonlinear parameters into modal coefficients results in the loss of the physical location of the respective nonlinear term. This point is conceded, although for the purposes of tractable aircraft prediction tools, the driving factor behind this research, the physical location of a discrete nonlinear element is incidental. Further, for continuous structures, the physical nonlinear response and ultimately the calculated HMT nonlinear FRFs is contingent upon the appropriate location of measurement transducers. A direct nonlinear identification approach is also presented based in part on the HMT method. Some limitations are noted; first, the nonlinear parameters are necessarily constant. The modified NIFO method presented herein is not constrained by this assumption, as the nonlinear coefficients in modal space are identified frequency-by-frequency. Further, Siller's nonlinear parameter identification method relies on a multi-step identification approach, where the measured response quantities are first divided into linear measured and nonlinear unmeasured quantities. Second, using the results of the initial identification, an assumed nonlinear form, e.g., cubic in the case of Siller's thesis, in physical space is then used to identify the constant nonlinear coefficients. Although little detail is provided as to the exact workings of the method, and in particular, details regarding the determination of the location of the nonlinear nodes in the FE model considered, the results are very promising albeit complicated. Finally, Siller makes a critical point, in that he also notes the risk in prematurely truncating what he terms 'non-essential data' or 'weak modes' as they can also have a significant affect on other modes via nonlinear coupling.

The analytical and experimental work discussed in this paper is based upon the previously published experimental study of Gordon et al. [31]. This experimental work considered the results of a well-characterized clamped–clamped beam as a means of exercising analytical nonlinear reduced order methods. Results of the published work include dynamic displacement and strain results for broadband random inputs. Additionally, experimental estimates of a cubic Duffing parameter for a SDOF model are also presented.

To date, there has been no published study of a truly practical and broadly applicable reduced order nonlinear identification method. There has been a significant amount of work recently in this area, but as previously discussed, the respective efforts require iterative schemes, hybrid modeling procedures, specific loading conditions, significant computational expense, multiple identification scenarios and generally complicated procedures. This effort presents a straightforward means of generating nonlinear reduced order models directly from raw experimental data utilizing the convenience of an analytically derived basis, although the actual source of the basis is incidental. The study is particularly important for the design and analysis of aircraft structures experiencing severe random acoustic loading. In this effort, both analytical as well as experimental studies were conducted on a well-characterized clamped–clamped beam setup. The experiment has been demonstrated to exhibit the form of nonlinear, amplitude dependent response typical of those structures experiencing acoustic fatigue. The displacement and acceleration data measured during the experiment was used to identify geometric nonlinear parameters. Analytical normal modes were utilized as the basis set necessary to filter or transform the experimental data from physical to reduced order space. Once the identification was accomplished, the nonlinear parameters were then used in the assembly of a nonlinear

MDOF reduced order model. The results of the model compared favorably with both analytical and experimental results, in both cases representing the beam at a more severe loading scenario.

2. Background

This nonlinear identification method was developed to determine a simple analytical model useful for prediction that adequately describes the behavior of an aircraft-like structure. The method is a reverse-path one, where raw experimental data along with an assumed mathematical model is utilized to identify nonlinear coefficients in reduced order space. Further, the method is parametric for the purposes of this investigation, although future work will investigate the use of non-parametric nonlinear models or trial functions. This method does not require an iterative scheme, nor are the individual mass, stiffness and damping parameters required or directly identified. What is required is a means of identifying a mass normalized basis set, either analytically or experimentally. The method is a spectral one, where nonlinear coefficients and the nominal linear system FRFs are identified in a least-squares sense. Further, the computational time required for the identification is negligible. The method is flexible, in that the proposed form of the nonlinearities can be easily modified and adapted to the problem at hand. Additionally, multiple nonlinearities of significantly different order can be identified simultaneously. The development of this method is based on the assumption that the nonlinear vibratory response of a structure can be adequately described by combinations of normal modes. As will be demonstrated, the nonlinear reduced order equations remain coupled; functions of the reduced order or modal coefficients.

The NIFO method of Adams and Allemang [32,33] was introduced as an effective means of identifying nonlinear structural parameters as well as the underlying, or nominally linear parameters in a single analysis step. One significant contribution of the method is the efficient use of spatial and temporal information in the characterization of nonlinear parameters. This use of spatial information allows for the identification of nonlinear parameters at both forced as well as unforced DOFs. Recall, this was also the impetus for the development of the CRP method [12]. The method introduced here is based upon the NIFO method, with one significant difference; the nonlinear dynamic equations used in the derivation are presented in a reduced order sense, and further, result in a mathematical representation of the structure of interest useful for prediction purposes. Consider Eq. (1), a representation of a general system equation of motion written in reduced order form:

$$\ddot{p}(t)_r + 2\xi_r\omega_r\dot{p}(t)_r + \omega_r^2 p(t)_r + \theta_r\big(p(t)_1, p(t)_2, \ldots, p(t)_n\big) = \phi_r^{\mathrm{T}} f(t), \tag{1}$$

where p_r represents the generalized or modal coordinate, ξ_r the modal damping, ω_r the natural frequencies, $f(t)$ the applied force in physical coordinates, ϕ_r the modal or basis vector, and θ_r the nonlinear modal components. Unlike a linear system, the equations represented in generalized coordinates are now coupled through the θ_r terms. It is important to point out that the appropriate nonlinear form of θ_r depends on the application in question. A general quadratic and cubic nonlinear model for n modes can be represented as

$$\theta_r = \sum_{i=1}^{n}\sum_{j=1}^{n} B_r(i,j)p(t)_i p(t)_j + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} A_r(i,j,k)p(t)_i p(t)_j p(t)_k. \tag{2}$$

In the present paper, the assumed form of the nonlinearities will be cubic based upon previous studies of aircraft structural response [6,7]. Further simplification was accomplished for the present work by assuming a two-mode cubic model, as the response for the structure currently under investigation can be adequately described using the first two symmetric bending modes. Thus, the nonlinear modal contribution for a cubic, two-mode model where in this case $n = 2$, takes the form

$$\theta_r = \sum_{i=1}^{n} A_r(i,i,i)p(t)_i^3 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\Big\{A_r(i,i,j)p(t)_i^2 p(t)_j + A_r(i,j,j)p(t)_i p(t)_j^2\Big\}, \tag{3}$$

where the A_r terms denote the nonlinear coefficients of interest. Applying the Fourier Transform to Eqs. (1) and (3) results in the modal impedance equation

$$B_r(\omega)P_r(\omega) + \Theta_r(\omega) = F_r(\omega), \tag{4}$$

where $B_r(\omega)$ is the linear modal impedance matrix, $P_r(\omega)$ is the rth modal coordinate in the frequency domain, $F_r(\omega)$ is the modal force in the frequency domain, and $\Theta_r(\omega)$ represents the nonlinear modal coupling terms transformed into the frequency domain. Recall that the Fourier Transform is based upon the assumption that the signal is a totally observed transient or consists solely of harmonics of the time period of observation.

Note that by using the system normal modes in the transformation from physical to modal space, the linear impedance matrix is uncoupled, while the $\Theta_r(\omega)$ terms remain coupled, as displayed in Eq. (3). Consider the following example where $r = 1$ in a two-mode expansion of Eq. (4), with the nonlinear terms moved to the RHS:

$$\begin{aligned} B_1(\omega)P_1(\omega) = F_1(\omega) &- A_1(1,1,1)\tilde{F}[p(t)_1^3] - A_1(2,2,2)\tilde{F}[p(t)_2^3] \\ &- A_1(1,1,2)\tilde{F}[p(t)_1^2 p(t)_2] - \cdots - A_1(1,2,2)\tilde{F}[p(t)_1 p(t)_2^2], \end{aligned} \tag{5}$$

where$\tilde{F}[\cdot]$ denotes the Fourier Transform. It is also convenient to rewrite Eq. (1) in physical space by introducing the respective physical DOF for the modal expansion term. For example, in this study the structural point of interest will be the beam midpoint or center. Therefore, the beam center transverse displacement can be represented as a function of the retained modes of interest:

$$x_{\text{center}} = \sum_{i=1}^{n} \phi_i^{\text{center}} p(t)_i. \tag{6}$$

Further, for the two-mode expansion example, considering only the displacement at the beam center, Eq. (6) can be represented as

$$x_{\text{center}} = x_1 + x_2 = \phi_1^{\text{center}} p(t)_1 + \phi_2^{\text{center}} p(t)_2. \tag{7}$$

Therefore, utilizing Eqs. (3) and (7), Eq. (1) with $r = 1$ can now be represented as

$$\begin{aligned} &\ddot{x}_1 + 2\xi_1\omega_1\dot{x}_1 + \omega_1^2 x_1 + \frac{A_1(1,1,1)}{\left(\phi_1^{\text{center}}\right)^2} x_1^3 + \frac{A_1(1,1,2)}{\phi_1^{\text{center}}\phi_2^{\text{center}}} x_1^2 x_2 + \frac{A_1(1,2,2)}{\left(\phi_2^{\text{center}}\right)^2} x_1 x_2^2 + \cdots + \frac{A_1(2,2,2)}{\left(\phi_2^{\text{center}}\right)^3}\phi_1^{\text{center}} x_2^3 \\ &= \phi_1^{\text{center}}\phi_1^{\text{T}} f(t). \end{aligned} \tag{8}$$

Note that in Eq. (8), the nonlinear coefficients are scaled by the respective basis set components. Continuing with the two-mode expansion example and pre-multiplying both sides of Eq. (5) by the linear *modal* FRF matrix, the following set of equations result:

$$\{P_1(\omega)\} = \begin{bmatrix} H_1(\omega) & H_1(\omega)A_1(1,1,1) & H_1(\omega)A_1(2,2,2) \\ \ldots H_1(\omega)A_1(1,1,2) & H_1(\omega)A_1(1,2,2) & \end{bmatrix} \left(\begin{matrix} \{F(\omega)\} \\ -\begin{pmatrix} \tilde{F}\left[p(t)_1^3\right] \\ \tilde{F}\left[p(t)_2^3\right] \\ \tilde{F}\left[p(t)_1^2 p(t)_2\right] \\ \tilde{F}\left[p(t)_1 p(t)_2^2\right] \end{pmatrix} \end{matrix} \right), \tag{9}$$

where the unknown nonlinear coefficients are post-multiplied by the measured responses. A similar equation in physical space results by utilizing Eq. (8) in the transformation to the frequency domain. In this instance, the $P_r(\omega)$ terms are replaced by the respective physical ones.

$$\{X_1(\omega)\} = \begin{bmatrix} H_1(\omega) & H_1(\omega)\hat{A}_1(1,1,1) & H_1(\omega)\hat{A}_1(2,2,2)\ldots \\ \ldots H_1(\omega)\hat{A}_1(1,1,2) & H_1(\omega)\hat{A}_1(1,2,2) & \end{bmatrix} \begin{pmatrix} \tilde{F}\{\phi_1^{\text{center}}\phi_1^{\text{T}}f(t)\} \\ -\begin{pmatrix} \tilde{F}\left[\frac{x(t)_1^3}{(\phi_1^{\text{center}})^2}\right] \\ \tilde{F}\left[\frac{x(t)_2^3}{(\phi_2^{\text{center}})^3}\phi_1^{\text{center}}\right] \\ \tilde{F}\left[\frac{x(t)_1^2 x(t)_2}{\phi_1^{\text{center}}\phi_2^{\text{center}}}\right] \\ \tilde{F}\left[\frac{x(t)_1 x(t)_2^2}{(\phi_2^{\text{center}})^2}\right] \end{pmatrix} \end{pmatrix}, \tag{10}$$

where in Eq. (10), the unknown nonlinear coefficients are now scaled by the appropriate basis set components. Note, $X_1(\omega)$ in Eq. (10) is not the physical response of the DOF 1, but the physical contribution of mode 1 to the total response as defined in Eq. (7). Since, experimental data is recorded in the physical domain, it will be necessary to transform or filter the data. This is accomplished via the following equation:

$$\{p(t)\} = \left[[\phi]^{\text{T}}[\phi]\right]^{-1}[\phi]^{\text{T}}\{x(t)\} = [\phi]^{+}\{x(t)\}, \tag{11}$$

where $[\phi]^{+}$ represents the Moore–Penrose pseudo-inverse of the proposed basis set, providing a best-fit solution in a least-squares sense.

Note that the measured nonlinear response is a function of the underlying linear system as well as the unmeasured nonlinear feedback forces, a tenet of the NIFO method. As previously discussed, the linear modal FRFs and nonlinear parameters can now be estimated in a single step via a least-squares formulation of Eqs. (9) or (10). The benefits of the modified NIFO method are now obvious: (1) estimates of the underlying linear modal FRF and nonlinear coefficients are achieved in a single-step versus the multi-step approaches of many of the previously described methods, (2) the method results in a series of SDOF linear modal FRFs, themselves appropriate for identification in modal space, (3) because NIFO is a frequency domain method, the linear and nonlinear portions of the equations are essentially uncoupled, and (4) the method is easily adaptable, thus providing researchers with a rapid means of attempting various nonlinear trial functions when various nonlinear elements are suspected or the appropriate form is unknown. The difficulty in obtaining accurate estimates of the nonlinear coefficients lies with the a priori knowledge of the form of the nonlinearity, $\Theta_r(\omega)$. Further, as the problem is a nonlinear one, it is understood that unique estimates cannot be obtained. What is sought is the best nonlinear modal model, useful for prediction purposes. Without explicit understanding of the form of the nonlinearities, general trial functions can be proposed, but without care, numerical issues can arise.

3. Analytical investigation

The analytical experiment was conducted on a FE model of a previously studied clamped–clamped beam. The clamped beam model considered in this investigation is displayed in Fig. 1. As previously mentioned, this beam model was developed specifically to exercise developing analytic prediction methods. This particular beam configuration was selected in the earlier referenced studies because it exhibits the form of nonlinear response typical of thin aircraft panel type structures. A 20-element symmetric model of the beam was created using nonlinear beam elements. Further, linear springs, the values of which were obtained through experimental characterization, were utilized to model the boundary conditions. The loading considered was broadband Gaussian random base excitation. By utilizing a flat input spectrum between 20 and 500 Hz, the first two symmetric bending modes of the beam were excited, 77.7 and 420 Hz, respectively. The dynamic response of the FE model was used as a truth model, as well as input to the modified NIFO method to recreate the beam model in reduced order space. A Newmark–Beta numerical integration technique, with a time step of 1.0e−4 s was used to generate the dynamic response. For each of five loading scenarios, 0.5, 1.0, 2.0, 4.0 and 8.0 g RMS, 120 s of displacement response was obtained. Proportional damping was introduced with damping values of 0.3% and 0.5% for the first two symmetric bending modes respectively. These proportional damping

Fig. 1. Analytic beam model with geometry and material properties. Length = 9 in; width = 0.5 in; thickness = 0.031 in; $E = 29.7$ Mpsi; $G = 11.6$ Mpsi; $\rho = 7.36 \times 10^{-4}\,\mathrm{lb\,s^2\,in^{-4}}$; $k_x = 2 \times 10^6\,\mathrm{lb\,in^{-1}}$; $k_y = 3 \times 10^6\,\mathrm{lb\,in^{-1}}$; $k_t = 1000\,\mathrm{in\,lb\,rad^{-1}}$.

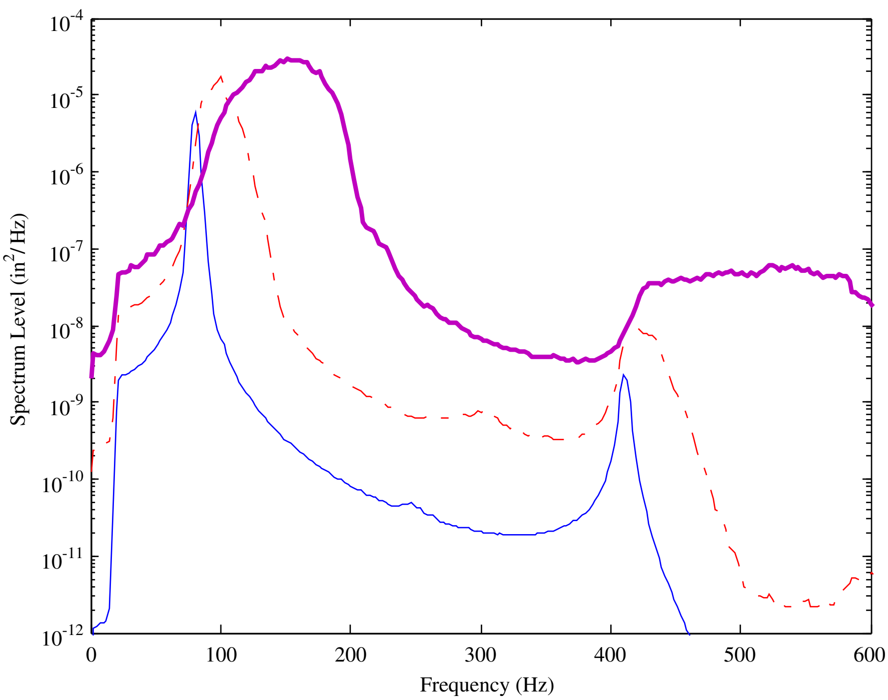

Fig. 2. Response of beam midpoint to random base-motion inertial loading (thin solid line: response to 0.5 g; dashed line: response to 2 g; bold solid line: response to 8 g).

values were obtained from experimental studies of the beam. The autopower spectrum or PSD of the beam midpoint is displayed in Fig. 2. Of particular note is the nonlinear response of the 2 and 8 g cases, as they will be used to identify the nonlinear modal parameters.

3.1. Analytical SDOF nonlinear parameter estimation

With the numerical experiment established, the dynamic displacement results were used to estimate and assemble the nonlinear modal models. Initially, a SDOF modal model was identified in order to directly compare with published values. It was found that using the 2 g load case resulted in the best spectral average of the hardening spring coefficient; an estimated value of $1.64\mathrm{e}+8\,\mathrm{in^{-2}\,s^{-2}}$ versus the published value of

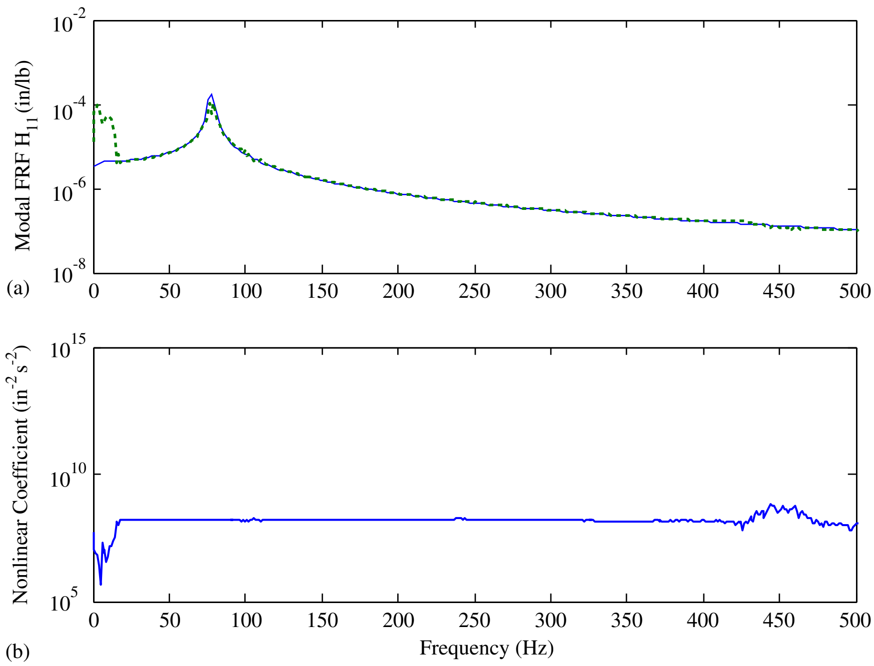

Fig. 3. SDOF estimates: (a) nominal linear FRF (solid line: low-level 'linear' FRF; bold dashed line: nominal linear FRF estimate), (b) cubic nonlinear coefficient.

$1.67e+8\,in^{-2}\,s^{-2}$. One interesting point was that using the higher load cases resulted in a 12% reduction in the value of the cubic coefficient when compared with the 'best' value estimated using the 2 g load case. The following parameters were used in the solution to and formulation of Eq. (10): 291 averages, a Hanning Window, 50% overlap and a blocksize of 4096. Fig. 3 displays the identified underlying modal FRF for the first bending mode, as well as the estimated nonlinear coefficient. Note that the coefficient is very nearly independent of frequency, although the nonlinear parameter does display some irregularities at very low frequency as well as between 400 and 500 Hz. The low-frequency discrepancy is due to the lack of prescribed input below 20 Hz. Conversely, the high-frequency irregularity denotes the influence of the second mode in the estimate. This is most likely due to a lack of spatial or measurement observability in the modal vectors used for the transformation from physical to modal space, an area that requires further study.

The 2-g load case was selected for use in Eq. (10) after evaluating the results using the response from each of the load scenarios, as well as the use of all of the response cases combined to form one contiguous time history. This underscores one potential pitfall of any nonlinear identification scheme, namely appropriate exercising of the nonlinearity. Without the ability to verify the identified parameters, it was beneficial to use a comparison of the measured and estimated nominal linear modal FRFs. In this numerical experiment, the linear modal FRF was arrived at by transforming the response of a very low level input into modal space—essentially a filtering of the beam midpoint response using the first bending mode shape. By directly comparing the FRFs, it was discovered that a poor FRF estimate was itself an indicator of poor nonlinear parameter estimates, a point to be studied further.

3.2. Analytical MDOF nonlinear parameter estimation

A 2-DOF modal model was identified using results from the 8 g loading scenario as input to Eq. (10). After reviewing estimates derived from the various response scenarios, the 8 g case was deemed the most appropriate, again based upon the nominal linear FRF estimate and resulting comparison with the analytical FRF. As explained in the lead-up to Eq. (5), a 2-DOF modal model, cubic in stiffness, will result in eight nonlinear coefficients. The identification is accomplished on a mode-by-mode basis, with four parameters

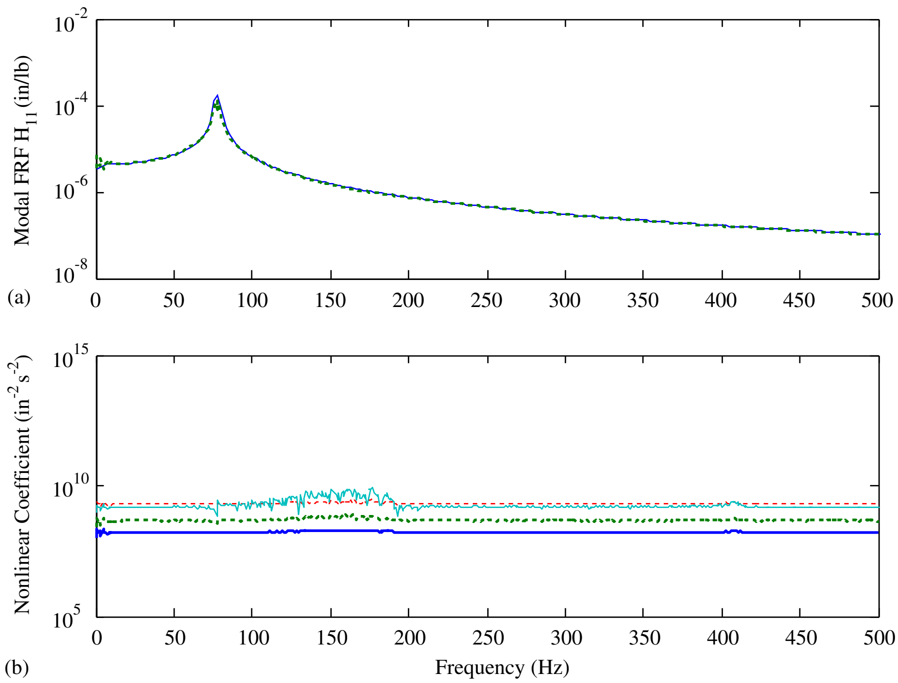

Fig. 4. MDOF Mode 1 estimates: (a) nominal linear FRF (solid line: low-level 'linear' FRF; bold dashed line: nominal linear FRF estimate), (b) cubic nonlinear coefficients (bold solid line: $A_1(1,1,1)$; bold dashed line: $A_1(1,1,2)$; solid line: $A_1(1,2,2)$; dashed line: $A_1(2,2,2)$).

identified for each mode, the A_r's in Eqs. (3) and (5). Similar frequency domain conditioning parameters were used for the MDOF case, namely 291 averages, a Hanning Window, 50% overlap and a blocksize of 4096. Graphical results of the estimation process are displayed in Figs. 4 and 5. Spectral averages of the estimated nonlinear coefficients along with values obtained using the *Direct Evaluation* [6] FE based identification method are presented in Table 1. The coefficients displayed in Table 1 represent the nonlinear coefficients in physical coordinates, as displayed in Eq. (10). In other words, they are scaled by the particular modal basis as described in Eq. (8). The estimates of both modes compare quite favorably with the results derived using the published method [6]. It is obvious, particularly comparing Figs. 4 and 5 that the estimates derived from the response of the second mode compare better with the actual nonlinear coefficients. Consider the nonlinear coefficient estimates in Fig. 5. There is negligible deviation from the average values displayed in Table 1, indicating a high confidence in the estimates. In contrast, consider the region near the first mode peak in Fig. 4, as well as the nonlinear coefficient estimates at nearly twice the fundamental frequency. Recall that for this study, only a cubic nonlinear form was assumed for θ_r. As was speculated in the SDOF estimate, it is also possible that further spatial discretization is required for the modal transformation. It is clear from the results of this study that enough measurement points be included for the modal vectors to properly filter the response data that generates the individual modal equations. For this particular study, all 21 nodal points and associated FE beam DOF were used in the transformation, a situation not possible with experimental analyses. The issue of spatial discretization is currently being investigated in an ongoing experimental effort. It is also important to point out that the response is dominated by the first mode. Thus, estimates of the second mode dominated nonlinear coefficients were not expected to be as accurate as those estimated for the first mode. The effect of the errors in the second mode dominated coefficients was insignificant, as will be demonstrated in the following discussion on prediction.

3.3. Analytical response prediction

Given the estimates of the nonlinear coefficients, a modal model was assembled and used for prediction purposes. In this study, only the MDOF results were used for prediction. Again, a Newmark–Beta numerical

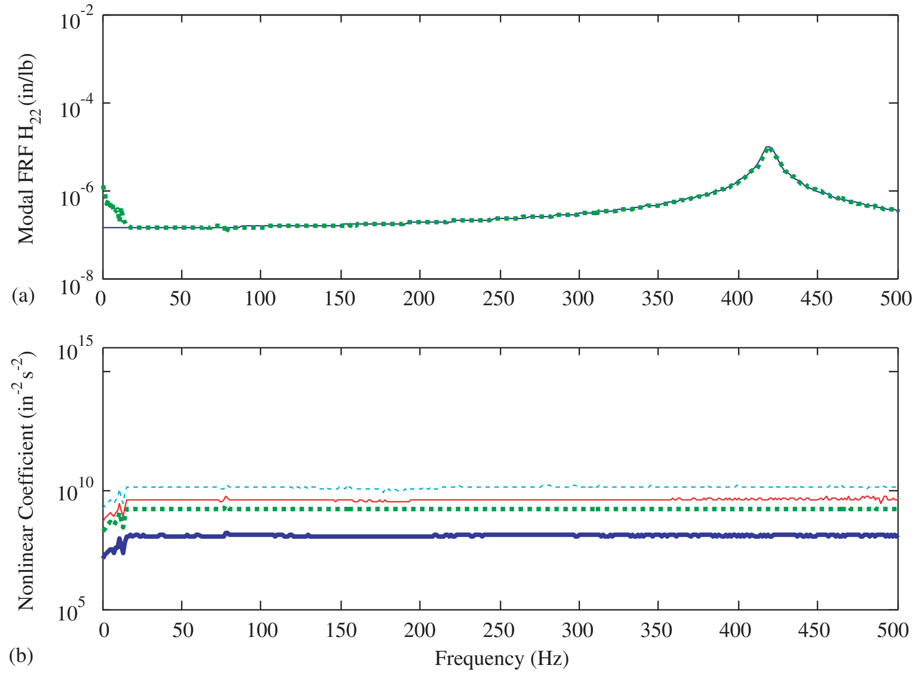

Fig. 5. MDOF Mode 2 estimates: (a) nominal linear FRF (solid line: low-level 'linear' FRF; bold dashed line: nominal linear FRF estimate), (b) cubic nonlinear coefficients (bold solid line: A_2(1,1,1); bold dashed line: A_2(1,1,2); solid line: A_2(1,2,2); dashed line: A_2(2,2,2)).

Table 1
2-DOF nonlinear estimates

Direct evaluation coefficients [6] ($in^{-2}s^{-2}$)	Estimated coefficients ($in^{-2}s^{-2}$)
$A_1(1,1,1) = 1.84e+8$	$A_1(1,1,1) = 1.76e+8$
$A_1(1,1,2) = 4.82e+8$	$A_1(1,1,2) = 5.17e+8$
$A_1(1,2,2) = 2.09e+9$	$A_1(1,2,2) = 2.13e+9$
$A_1(2,2,2) = 1.58e+9$	$A_1(2,2,2) = 2.58e+9$
$A_2(1,1,1) = 1.26e+8$	$A_2(1,1,1) = 1.14e+8$
$A_2(1,1,2) = 1.64e+9$	$A_2(1,1,2) = 1.52e+9$
$A_2(1,2,2) = 3.73e+9$	$A_2(1,2,2) = 3.54e+9$
$A_2(2,2,2) = 1.39e+10$	$A_2(2,2,2) = 1.35e+10$

integration scheme with a time-step of 1.0e−4 s was used. It is important to point out that calculating 120 s of data using the aforementioned modal models required approximately 300 s versus the 2–4 h for comparable full-model, direct integration of this particular and relatively simple FE beam model. Thus, one of the benefits of obtaining reduced order models is obvious—significant computational savings. Results of the prediction for the 4 and 8 g input load cases are displayed in Fig. 6. Recall, the 8 g load scenario was used for the MDOF parameter identification. Further, RMS values of the response for all of the loading cases are presented in Table 2. Again, broadband random input excitation was used with the MDOF estimated modal model. Both the displacement PSD and RMS values clearly indicate excellent agreement with the analytic results.

4. Experimental investigation

A high-fidelity experiment was conducted using a configuration similar to that presented by Gordon et al. [31]. The test setup for the current effort is presented in Fig. 7. Notice that only half of the beam

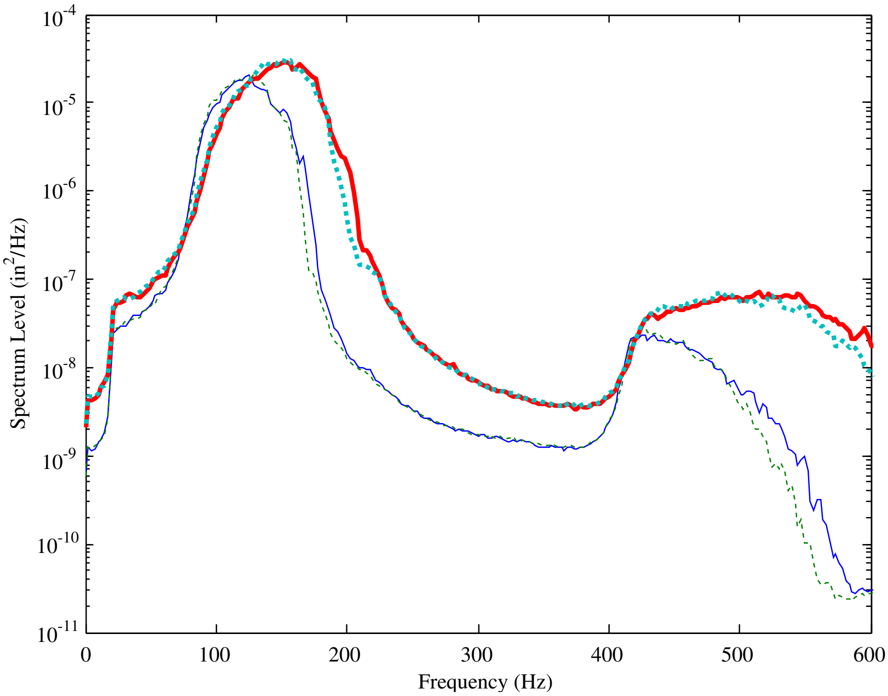

Fig. 6. Analytical prediction using identified nonlinear coefficients (solid line: 4 g analytic load case; dashed line: 4g predicted load case; bold solid line: 8 g analytic load case; bold dashed line: 8 g predicted load case).

Table 2
RMS displacement values

Load (g)	Experimental results (in)	Prediction using estimated coefficients (in)
0.5	0.006	0.007
1.0	0.012	0.012
2.0	0.021	0.019
4.0	0.031	0.031
8.0	0.044	0.043

was instrumented. This issue will be addressed subsequently. As previously discussed, this particular experimental setup was designed to exhibit the type of nonlinear response typical of aircraft structures experiencing sonic fatigue. The nominal beam dimensions and material properties are displayed in Fig. 1. Inertial loading was applied to the beam using a 1200-lb shaker with the shaker oriented in such a way as to mitigate the influence of gravity on the beam. In order to maintain the desired spectral content and RMS input level, a closed-loop controller was used via an accelerometer mounted to the shaker head. There were six measurement locations along the beam; acceleration was recorded at five locations using micro-accelerometers, and a laser vibrometer was used to record displacement at the beam midpoint. The accelerometer measurements were integrated twice in the frequency domain to obtain the respective displacement time histories. Throughout the testing, a sampling rate of 4096 Hz was used, and test record lengths of 48 s were recorded. Just as in the analytical experiment, a broadband random input between 20 and 500 Hz was used for all of the loading scenarios. This loading spectrum allowed for the first two bending modes of the beam to be excited, and is a spectrum typical of that experienced by aircraft structures in the presence of engine induced random loading.

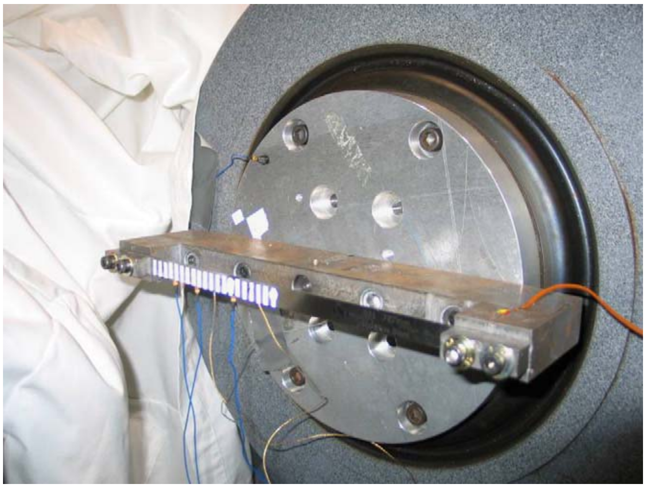

Fig. 7. Clamped–clamped beam experimental setup.

4.1. Experimental SDOF nonlinear parameter estimation

A useful nonlinear characterization test [31] was conducted prior to the start of the random vibration testing and nonlinear identification. This characterization test is a combination of several nonlinear SDOF identification algorithms [34–36], the intent of which is to identify an equivalent SDOF cubic coefficient via the backbone curve for the Duffing equation. This is important, as the test will provide one of the few metrics to compare with the proposed modified NIFO method. Note that the previously discussed analytical identification results had the benefit of published coefficients with which to compare. With the exception of the aforementioned characterization test, the true proof of the method will be the results of the prediction study based on the experimental identification. In addition to the cubic coefficient, the linear natural frequency and damping can also be identified using this simple test. The characterization begins by giving the beam an initial displacement at or near the beam midpoint and allowing the beam to decay freely. A Hilbert transform is then applied to segments of the recorded time history, and amplitude and frequency estimates of the transformed time segments are used to assemble the backbone curve. The cubic coefficient was then identified as the slope of the plot of the identified frequency squared versus beam amplitude squared. The Duffing coefficient is independent of small variations in beam temperature [31]. This is important because a slight tensile preload was applied to the beam as the clamping fixtures were tightened in order to prevent the beam from buckling. Further, it was previously demonstrated that small variations in the beam temperature could be appropriately captured in an FE model through changes to the linear stiffness term/s. Using the previously described SDOF free-decay characterization test, the Duffing coefficient was estimated to be $1.12\mathrm{e}+8\,\mathrm{in}^{-2}\,\mathrm{s}^{-2}$. The results of the SDOF identification using the modal NIFO approach are presented in Fig. 8, in the vicinity of the first mode. Note that there are negligible differences between the linear modal FRFs. The differences that exist are likely due to any nominal preload applied during the test setup. It is important to point out that the nominal linear modal FRF was obtained from the response of the beam at very low level testing. This configuration will be used for the MDOF identification as well.

4.2. Experimental MDOF nonlinear parameter estimation

The real benefit of the proposed nonlinear identification approach, in addition to its ease of use and flexibility in assuming various nonlinear forms, is the extraction of multiple nonlinear parameters at a single time. While numerous SDOF nonlinear identification algorithms currently exist as previously discussed, the same cannot be said regarding a robust and proven MDOF reduced order method. For the MDOF case, the

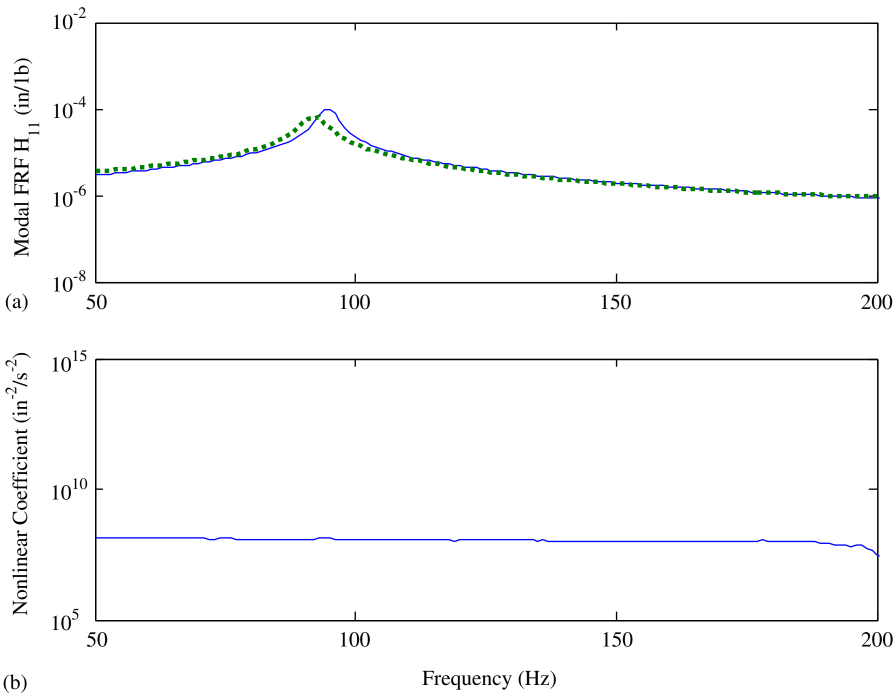

Fig. 8. SDOF estimates: (a) nominal linear FRF (solid line: low-level 'linear' FRF; bold dashed line: nominal linear FRF estimate), (b) cubic nonlinear coefficient.

first two symmetric beam bending modes were used to filter the raw experimental test data. This is a trivial exercise for the SDOF scenario, but not for the MDOF one. Some rules must be followed when filtering the data with a normal mode basis set. First, the data must be synchronous or recorded at the same time. The use of synchronous data allows for accurate capturing of the relative motion of the sensors. The sensor locations indicated by the asterisks and their position relative to the basis set are displayed in Fig. 9. The results of the successful filtering using the two symmetric normal modes displayed in Fig. 9 are now displayed in Fig. 10. Note the significant difference in amplitude between the two filtered time histories. Further, note the differences in the spectral content, particularly at the first mode. The two PSDs represent the spectral content of the respective modal time histories. The ability of the proposed modal NIFO method to identify a series of nonlinear coefficients along with the nominal linear modal FRFs will be demonstrated to be quite powerful. One other issue currently being investigated is the most appropriate spatial discretization, as well as the most appropriate number of sensors in order to achieve optimal filtering.

Applying the modal filtering called out in Eq. (11) using the first two symmetric beam modes and conducting the MDOF identification called out in Eq. (10), resulted in the two nominally linear FRFs and the eight nonlinear cubic parameters displayed graphically in Figs. 11 and 12. The spectral averages of the nonlinear coefficients are displayed in Fig. 13. It should be noted that the values displayed in Figs. 11 and 12 represent the real parts of the nonlinear coefficients. Note that the imaginary components of the identified nonlinear parameters were minimal, at least an order of magnitude less than the respective real components, and therefore not represented in the results. Also note that the identified coefficients are not a function of frequency. Although there was no direct means to compare the eight identified coefficients, observe the first mode cubic parameter, $1.27\mathrm{e}+8\,\mathrm{in}^{-2}\,\mathrm{s}^{-2}$. The identified coefficient is nominally 13% greater than both the SDOF experimental case as well as the free-decay test described earlier. It is theorized that the difference is due to the inclusion of additional nonlinear parameters. Further, note the good approximation of the nominal linear modal FRFs, particularly for the second mode, in the vicinity of the respective modes.

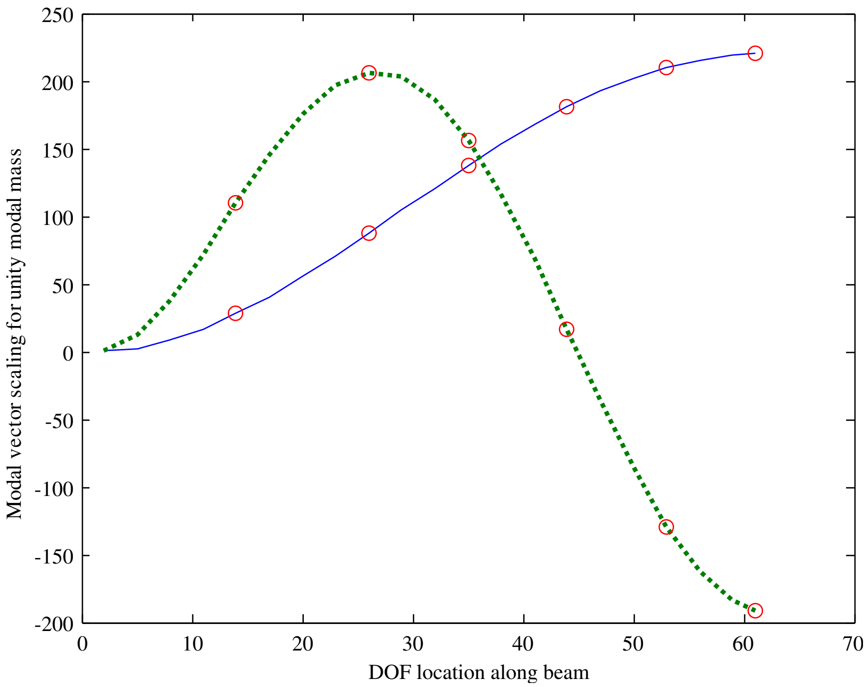

Fig. 9. Symmetric beam bending modes and measurement locations used for modal filtering (solid line: 1st symmetric bending mode; bold dashed line: 2nd symmetric bending mode; asterisks denote experimental sensor measurement locations).

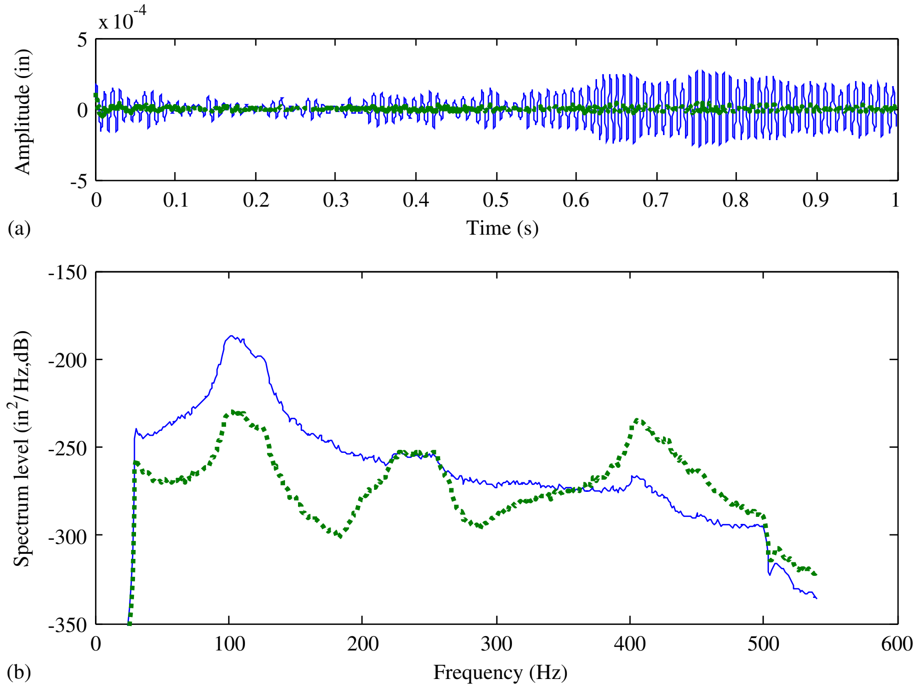

Fig. 10. Modal filtering of experimental 4 g RMS test case: (a) representative modal time history, (b) filtered modal spectra. (solid line: experimental time history filtered via 1st mode; bold dashed line: experimental time history filtered via 2nd mode).

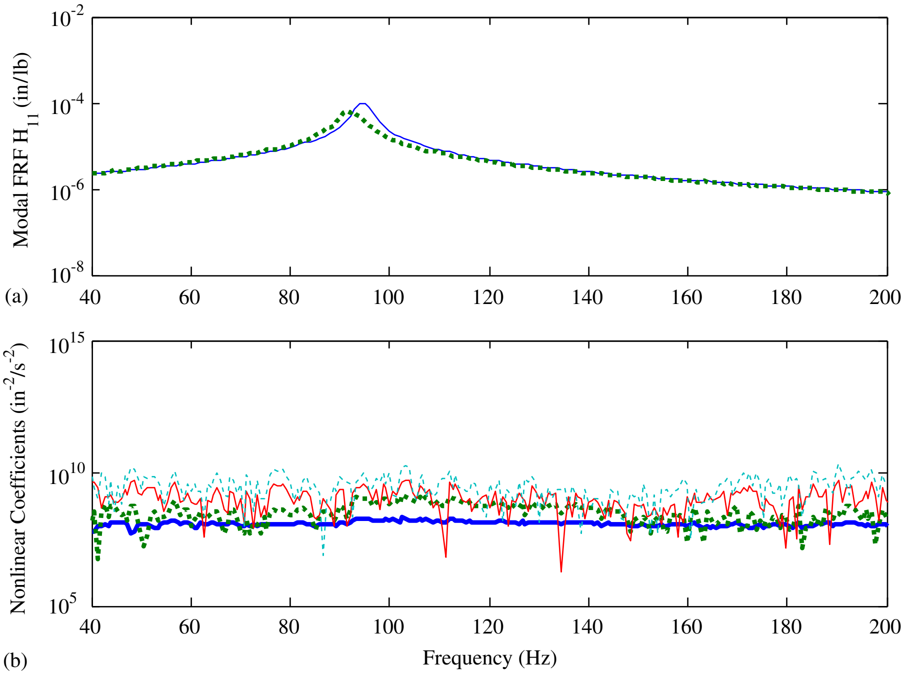

Fig. 11. Experimental MDOF Mode 1 estimates: (a) nominal linear FRF (solid line: low-level 'linear' FRF; bold dashed line: nominal linear FRF estimate), (b) cubic nonlinear coefficients (bold solid line: $A_1(1,1,1)$; bold dashed line: $A_1(1,1,2)$; solid line: $A_1(1,2,2)$; dashed line: $A_1(2,2,2)$).

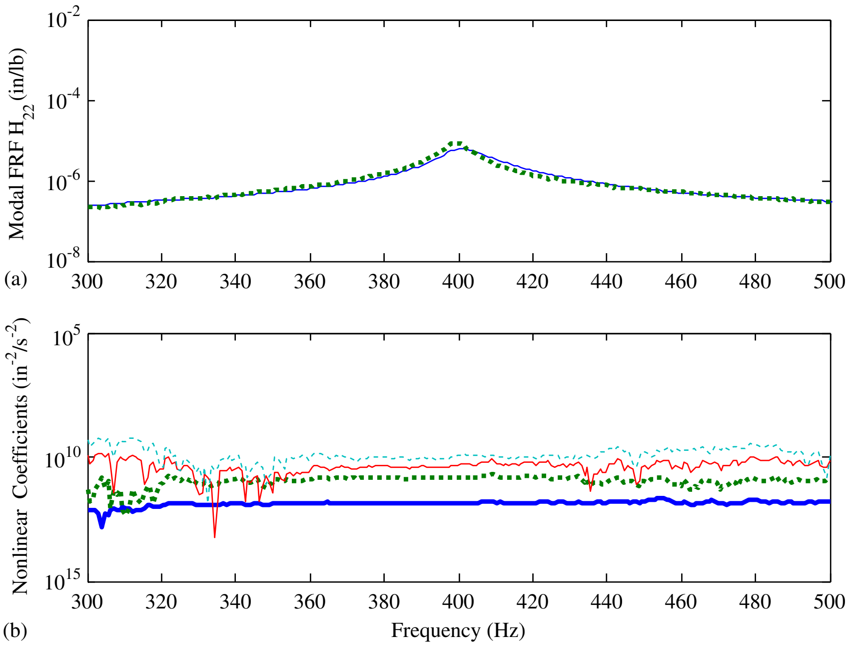

Fig. 12. Experimental MDOF Mode 2 estimates: (a) nominal linear FRF (solid line: low-level 'linear' FRF; bold dashed line: nominal linear FRF estimate), (b) cubic nonlinear coefficients (bold solid line: $A_2(1,1,1)$; bold dashed line: $A_2(1,1,2)$; solid line: $A_2(1,2,2)$; dashed line: $A_2(2,2,2)$).

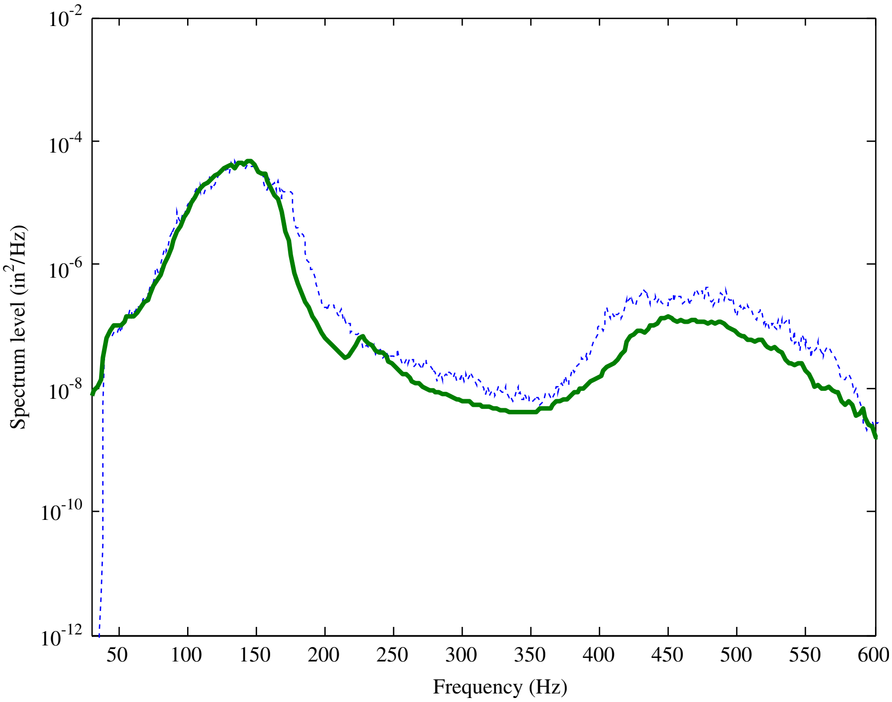

Fig. 13. Experimental prediction of 8 g RMS loading using identified nonlinear coefficients. $A_1(1,1,1) = 1.27\text{e}+8\,\text{in}^{-2}\,\text{s}^{-2}$; $A_1(1,1,2) = 4.12\text{e}+8\,\text{in}^{-2}\,\text{s}^{-2}$; $A_1(1,2,2) = 1.38\text{e}+9\,\text{in}^{-2}\,\text{s}^{-2}$; $A_1(2,2,2) = 4.02\text{e}+9\,\text{in}^{-2}\,\text{s}^{-2}$; $A_2(1,1,1) = 1.49\text{e}+8\,\text{in}^{-2}\,\text{s}^{-2}$; $A_2(1,1,2) = 1.27\text{e}+9\,\text{in}^{-2}\,\text{s}^{-2}$; $A_2(1,2,2) = 4.72\text{e}+9\,\text{in}^{-2}\,\text{s}^{-2}$; $A_2(2,2,2) = 1.35\text{e}+10\,\text{in}^{-2}\,\text{s}^{-2}$. (Dashed line: experimental spectrum; bold solid line: prediction).

4.3. Experimental MDOF nonlinear parameter estimation

The real value in the proposed identification method is the ability to conduct predictions based upon the assembled modal model. Fig. 13 displays the results of such a prediction. The assembled nonlinear reduced order model is compared with experimental data representing twice the load (8 versus 4 g RMS) used for the identification. The response PSD of the beam midpoint displayed in Fig. 13 equates to RMS values of the estimate and experiment beam midpoint displacement of 0.045 versus 0.047 in respectively. The spectral comparisons are also quite good, particularly for the first mode. Note the appearance of a peak in the 200 Hz range of the 'estimated' PSD. It is not clear whether the unknown peak is an artifact of the numerical integration, or one associated with the identification itself, i.e., the most appropriate model given the influence of the sensors. As previously mentioned, another issue under investigation is the number of sensors and the appropriate positioning. Recall that for this test, the accelerometers were positioned on only one side of the beam. Further, the modal mass of the beam relative the sensor mass will also play a significant role for such a lightly damped, light-weight structure. An optimal testing configuration would be laser vibrometers appropriately spaced along the beam, but this was not a viable option at the time of testing.

5. Conclusion and future work

A novel approach was presented for the estimation and generation of nonlinear modal models. Accurate estimates were obtained for both SDOF and MDOF analytical and experimental modal models. Excellent agreement was noted between the analytical identification and previously published results. Further, the MDOF modal model was used for prediction purposes, again with favorable comparisons. Reduced order models were also assembled using experimental data. A SDOF reduced order model was compared with a free-decay nonlinear identification test method. The Duffing estimates compare well with the estimates gleaned from the method. Further, good RMS and spectral comparisons were made between experimental and the modified NIFO based reduced order model, at a significantly higher input scenario. The method presented in this study is quite flexible in that it is a simple matter to adjust the assumed form of the nonlinearity. As the end result is a reduced order model, researchers have the ability to quickly and accurately predict the design space. Future work will investigate the issue of spatial truncation in developing experimental modal vectors, as well as the issues associated with closely spaced modes. It is expected that the developing tool will be of great benefit for the analysis of highly complex structures in extreme combined environments.

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